

Self-similarity and fractional Brownian motions on Lie groups

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Abstract

The goal of this paper is to define and study a notion of fractional Brownian motion on a Lie group. We define it as the solution of a stochastic differential equation driven by a linear fractional Brownian motion. We show that this process has stationary increments and satisfies a local self-similar property. Furthermore the Lie groups for which this self-similar property is global are characterized. Finally, we prove an integration by parts formula on the path group space and deduce the existence of a density.

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1 Introduction

Since the seminal works of Itô [10], Hunt [9], and Yosida [26], it is well known that the (left) Brownian motion on a Lie group \mathbf{G} appears as the solution of a stochastic differential equation

$$dX_t = \sum_{i=1}^d V_i(X_t) \circ dB_t^i, \quad t \geq 0, \quad (1.1)$$

where V_1, \dots, V_d are left-invariant vector fields on \mathbf{G} and where $(B_t)_{t \geq 0}$ is a Brownian motion; for further details on this, we also refer to [1] and [23]. In this paper we investigate the properties of the solution of an equation of the type (1.1) when the driving Brownian motion is replaced by a fractional Brownian motion with parameter H . Recently, there have been numerous attempts to define a notion of solution for differential equations driven by fractional Brownian motion. One dimensional differential equation can be solved using a Doss-Susmann approach for any values of the parameter H as in the work of Nourdin, [19] or in the linear case by Nourdin-Tudor in [20]. The situation is quite different in the multidimensional case. When the Hurst parameter is greater than $1/2$ existence and uniqueness of the solution are obtained by Zähle in [28] or Nualart-Rascânu in [21]. And finally, as a consequence of the work of Coutin and Qian [7] a notion of solution is actually well-defined for $H > \frac{1}{4}$.

The paper is organized as follows.

In a first section, we show existence and uniqueness for the solution of an equation of the type (1.1) when $(B_t)_{t \geq 0}$ is a fractional Brownian motion with parameter $H > \frac{1}{4}$. The solution is shown to have stationary increments. We also check that the solution is invariant in law by isometries.

In the second section, we study the scaling properties of the solution. In the spirit of the notion of asymptotic self-similarity studied by Kunita [11], [12], (see also [5]), we show that the fractional Brownian motion on the group is asymptotically self-similar with parameter H . After that, we characterize the groups for which the scaling property is global: such groups are necessarily simply connected and nilpotent.

Finally, the goal of the last section is to show the existence of a density for the solution. At this end, we prove an integration by parts formula on the path group space.

To simplify the presentation of our results, we mainly worked in the setting of Lie groups of matrices. Nevertheless all our results extend to general Lie groups.

2 Fractional Brownian motion on a Lie group

Let us first recall that a d -dimensional fractional Brownian motion with Hurst parameter $H \in (0, 1)$ is a Gaussian process

$$B_t = (B_t^1, \dots, B_t^d), \quad t \geq 0,$$

where B^1, \dots, B^d are d independent centered Gaussian processes with covariance function

$$R(t, s) = \frac{1}{2} (s^{2H} + t^{2H} - |t - s|^{2H}).$$

It can be shown that such a process admits a continuous version whose paths are Hölder p continuous, $p < H$. Let us observe that for $H = \frac{1}{2}$, B is a Brownian motion.

Let \mathbf{G} be a finite-dimensional ($\dim \mathbf{G} = d$) connected Lie group of matrices with Lie algebra \mathfrak{g} . We consider a basis (V_1, \dots, V_d) of \mathfrak{g} .

If $(B_t^1, \dots, B_t^d)_{t \geq 0}$ is a d -dimensional fractional Brownian motion in \mathbb{R}^d with Hurst parameter $H \in (0, 1)$. The process

$$B_t^{\mathfrak{g}} = \sum_{i=1}^d B_t^i V_i$$

shall be called the canonical fractional Brownian motion on \mathfrak{g} with respect to the basis (V_1, \dots, V_d) .

In the remainder of this section, we assume now $H > \frac{1}{4}$.

Theorem 2.1 *The equation*

$$dX_t = X_t dB_t^{\mathfrak{g}}, \quad X_0 = 1_{\mathbf{G}} \tag{2.2}$$

has a unique solution in \mathbf{G} in the sense of rough paths of [15]. This solution $(X_t)_{t \geq 0}$ satisfies for every $s \geq 0$, $(X_s^{-1} X_{t+s})_{t \geq 0} =^{law} (X_t)_{t \geq 0}$. The process $(X_t)_{t \geq 0}$ shall be called a left fractional Brownian motion with parameter H on \mathbf{G} with respect to the basis (V_1, \dots, V_d) .

Proof. We first show the existence and the uniqueness of a solution in \mathbf{G} . Without loss of generality, we can work on the time interval $[0, 1]$. First, according to [7], the equation (2.2) has a unique solution with finite p variation for $p > \frac{1}{H}$. Secondly, let $B^{\mathfrak{g}, m}$ be the sequel of linear interpolation of $B^{\mathfrak{g}}$ along the dyadic subdivision of mesh m ; that is if $t_i^m = i2^{-m}$ for $i = 0, \dots, 2^m$; then for $t \in [t_i^m, t_{i+1}^m)$,

$$B_t^{\mathfrak{g}, m} = B^{\mathfrak{g}}(t_i^m) + \frac{t - t_i^m}{t_{i+1}^m - t_i^m} (B_{t_{i+1}^m}^{\mathfrak{g}} - B_{t_i^m}^{\mathfrak{g}}).$$

Let us now denote X^m the solution of (2.2) where $B^{\mathfrak{g}}$ is replaced by $B^{\mathfrak{g}, m}$, that is

$$dX_t^m = \sum_{k=1}^d X_t^m dB_t^{k, m} V_k$$

It is easily seen that for $t \in [t_{n-1}^m, t_n^m)$, recursively on n , $n = \dots, 2^m - 1$,

$$X_t^m = \exp \left(2^m (t - t_{n-1}^m) \sum_{k=1}^d (B_{t_n^m}^k - B_{t_{n-1}^m}^k) V_k \right) \cdots \exp \left(\sum_{k=1}^d (B_{t_1^m}^k - B_{t_0^m}^k) V_k \right).$$

Therefore, X^m takes its values in \mathbf{G} . Now, from [7], X^m converges to X for the distance of $1/p$ Hölder. Since the group \mathbf{G} is closed in the $1/p$ Hölder topology, we conclude that X belongs to \mathbf{G} .

We now show that for every $s \geq 0$, the processes $((X_s)^{-1} X_{t+s}, t \geq 0)$ and $(X_t, t \geq 0)$ have the same law. Let us fix $s \geq 0$. Once time again, the idea is to use a linear interpolation along the dyadic subdivision of $[0, 1]$ of mesh m and we keep the previous notations. First, let us observe that for $t \geq s$,

$$X_t = X_s + \int_s^t X_u dB_u^{\mathfrak{g}}. \quad (2.3)$$

Let us now denote $(X_t^{m,s})_{s \leq t \leq s+1}$ the solution of (2.3) where $B^{\mathfrak{g}}$ is replaced by $(B_{s+t}^{\mathfrak{g},m})_{0 \leq t \leq 1}$. Therefore, for $t \in [t_{n-1}^m, t_n^m)$,

$$X_{t+s}^{m,s} = X_s \exp \left(2^m (t - t_{n-1}^m) \sum_{k=1}^d (B_{s+t_n^m}^k - B_{s+t_{n-1}^m}^k) V_k \right) \cdots \exp \left(\sum_{k=1}^d (B_{s+t_1^m}^k - B_{s+t_0^m}^k) V_k \right).$$

By using the stationarity of the increments of the Euclidean fractional Brownian motion, we get therefore:

$$(X_s^{-1} X_{t+s}^{m,s})_{0 \leq t \leq 1} \stackrel{\text{law}}{=} (X_t^m)_{0 \leq t \leq 1}$$

Using the Wong-Zakai theorem of [7] and passing to the limit, we obtain that for every $s \geq 0$, $(X_s^{-1} X_{t+s})_{t \geq 0} \stackrel{\text{law}}{=} (X_t)_{t \geq 0}$. \square

Remark 2.2 *In the same way, we call the solution of the differential equation*

$$dX_t = dB_t^{\mathfrak{g}} X_t, \quad X_0 = \mathbf{1}_{\mathbf{G}}.$$

a right fractional Brownian motion on \mathbf{G} . It is easily seen that if $(X_t)_{t \geq 0}$ is a left fractional Brownian motion on \mathbf{G} , then $(X_t^{-1})_{t \geq 0}$ is a right fractional Brownian motion on \mathbf{G} .

Let us now turn to some examples.

Example 2.3 *The first basic example is $(\mathbf{R}^d, +)$. In that case, the Lie algebra is generated by the vector fields $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}$ and the fractional Brownian on $(\mathbf{R}^d, +)$ is nothing else but the usual Euclidean fractional Brownian motion.*

Example 2.4 *The second basic example is the circle. Let*

$$\mathbf{S}^1 = \{z \in \mathbb{C}, |z| = 1\}.$$

The Lie algebra of \mathbf{S}^1 is \mathbb{R} and is generated by $\frac{\partial}{\partial \theta}$ and the fractional Brownian motion on \mathbf{S}^1 is given by

$$X_t = e^{iB_t}, \quad t \geq 0,$$

where $(B_t)_{t \geq 0}$ is a fractional Brownian motion on \mathbb{R} .

Example 2.5 Let us consider the Lie group $\mathbf{SO}(3)$, i.e. the group of 3×3 , real, orthogonal matrices of determinant 1. Its Lie algebra $\mathfrak{so}(3)$ consists of 3×3 , real, skew-adjoint matrices of trace 0. A basis of $\mathfrak{so}(3)$ is formed by

$$V_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad V_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

A left fractional Brownian motion on $\mathbf{SO}(3)$ is therefore given by the solution of the linear equation

$$dX_t = X_t \begin{pmatrix} 0 & dB_t^1 & dB_t^3 \\ -dB_t^1 & 0 & dB_t^2 \\ -dB_t^3 & -dB_t^2 & 0 \end{pmatrix}, \quad X_0 = 1.$$

This notion of fractional Brownian motion on a Lie group is invariant by isometries, so that the law is invariant by an orthonormal change of basis. More precisely, let us consider the scalar product on \mathfrak{g} that makes the basis V_1, \dots, V_d orthonormal. This scalar product defines a Riemannian structure on \mathbf{G} for which the left action is an action by isometries. We have the following proposition:

Proposition 2.6 Let $\Psi : \mathbf{G} \rightarrow \mathbf{G}$ be a Lie group morphism such that $d\Psi_{1_{\mathbf{G}}}$ (differential of Ψ at $1_{\mathbf{G}}$) is an isometry and let $(X_t)_{t \geq 0}$ be the left fractional Brownian motion on \mathbf{G} as defined in Theorem 2.1. We have:

$$(\Psi(X_t))_{t \geq 0} =^{law} (X_t)_{t \geq 0}.$$

Proof. Let us observe that by the change of variable formula, see [16]:

$$d\Psi(X_t) = \Psi(X_t) \left(\sum_{i=1}^d d\Psi_{1_{\mathbf{G}}}(V_i) B_t^i \right),$$

Now,

$$\left(\sum_{i=1}^d d\Psi_{1_{\mathbf{G}}}(V_i) B_t^i \right)_{t \geq 0} =^{law} \left(\sum_{i=1}^d V_i B_t^i \right)_{t \geq 0},$$

because of the orthogonal invariance of the Euclidean fractional Brownian motion. Therefore,

$$(\Psi(X_t))_{t \geq 0} =^{law} (X_t)_{t \geq 0}.$$

□

Remark 2.7 If \mathbf{G} is compact then there exists a bi-invariant Riemannian metric and so, if $(X_t)_{t \geq 0}$ denotes a left fractional Brownian motion for this bi-invariant metric, from the previous proposition, we get that for every $g \in \mathbf{G}$,

$$(gX_tg^{-1})_{t \geq 0} =^{law} (X_t)_{t \geq 0}.$$

If the group \mathbf{G} is nilpotent then we have a closed formula for the left fractional Brownian motion on \mathbf{G} that extends the well-known formula for the Brownian motion on a nilpotent group (see by e.g. [1], [4] or [25]).

Let us introduce some notations: For $k \geq 1$,

•

$$\Delta^k[0, t] = \{(t_1, \dots, t_k) \in [0, t]^k, t_1 < \dots < t_k\};$$

- If $I = (i_1, \dots, i_k) \in \{1, \dots, d\}^k$ is a word with length k ,

$$\int_{\Delta^k[0, t]} dB^I = \int_{0 < t_1 < \dots < t_k \leq t} dB_{t_1}^{i_1} \dots dB_{t_k}^{i_k};$$

- We denote \mathfrak{S}_k the group of the permutations of the index set $\{1, \dots, k\}$ and if $\sigma \in \mathfrak{S}_k$, we denote for a word $I = (i_1, \dots, i_k)$, $\sigma \cdot I$ the word $(i_{\sigma(1)}, \dots, i_{\sigma(k)})$;
- If $I = (i_1, \dots, i_k) \in \{1, \dots, d\}^k$ is a word, we denote by V_I the Lie commutator defined by

$$V_I = [V_{i_1}, [V_{i_2}, \dots, [V_{i_{k-1}}, V_{i_k}] \dots]];$$

- If $\sigma \in \mathfrak{S}_k$, we denote $e(\sigma)$ the cardinality of the set

$$\{j \in \{1, \dots, k-1\}, \sigma(j) > \sigma(j+1)\};$$

- Finally, if $I = (i_1, \dots, i_k) \in \{1, \dots, d\}^k$ is a word

$$\Lambda_I(B)_t = \sum_{\sigma \in \mathfrak{S}_k} \frac{(-1)^{e(\sigma)}}{k^2 \binom{k-1}{e(\sigma)}} \int_{\Delta^k[0, t]} \circ dB^{\sigma^{-1} \cdot I}.$$

Proposition 2.8 Assume that \mathbf{G} is a nilpotent group then:

$$X_t = \exp \left(\sum_{k=1}^{+\infty} \sum_{I=(i_1, \dots, i_k)} \Lambda_I(B)_t V_I \right), \quad t \geq 0,$$

where the above sum is actually finite and where $(X_t)_{t \geq 0}$ is the left fractional Brownian motion defined as in Theorem 2.1.

Proof. Let B^m be the sequel of linear interpolation of B^\natural along the dyadic subdivision of mesh m . Let us now denote X^m the solution of (2.2) where B is replaced by B^m . As already seen, for $t \in [t_{n-1}^m, t_n^m)$,

$$X_t^m = \exp \left(2^m (t - t_{n-1}^m) \sum_{k=1}^d (B_t^k - B_{t_{n-1}^m}^k) V_k \right) \cdots \exp \left(\sum_{k=1}^d (B_{t_1^m}^k - B_{t_0^m}^k) V_k \right).$$

Now we use the Baker-Campbell-Hausdorff formula in nilpotent Lie groups (see [1], [4], [24]) to write the previous product of exponentials under the form

$$X_t^m = \exp \left(\sum_{k=1}^{+\infty} \sum_{I=\{i_1, \dots, i_k\}} \Lambda_I(B^m)_t V_I \right),$$

where

$$\Lambda_I(B^m)_t = \sum_{\sigma \in \mathfrak{S}_k} \frac{(-1)^{e(\sigma)}}{k^2 \binom{k-1}{e(\sigma)}} \int_{\Delta^k[0,t]} \circ dB^{m, \sigma^{-1} \cdot I}.$$

From [7], in the distance of p variation, with $p > \frac{1}{H}$, and if the length of the word I is less than 2,

$$\Lambda_I(B^m)_t \rightarrow_{m \rightarrow +\infty} \Lambda_I(B)_t. \quad (2.4)$$

By using now Theorem 3.1.3 of [16], the convergence in (2.4) holds for all word. Therefore,

$$X_t = \exp \left(\sum_{k=1}^{+\infty} \sum_{I=(i_1, \dots, i_k)} \Lambda_I(B)_t V_I \right), \quad t \geq 0.$$

□

Example 2.9 *In a two-step nilpotent group, that if is all brackets with length more than two are zero, we have therefore*

$$X_t = \exp \left(\sum_{i=1}^d B_t^i V_i + \frac{1}{2} \sum_{1 \leq i < j \leq d} \left(\int_0^t B_s^i dB_s^j - B_s^j dB_s^i \right) [V_i, V_j] \right).$$

3 Self-similarity of a fractional Brownian motion on a Lie group

Recall that for the Euclidean fractional Brownian motion, we have

$$(B_{ct}^1, \dots, B_{ct}^d)_{t \geq 0} \stackrel{law}{=} (c^H B_t^1, \dots, c^H B_t^d)_{t \geq 0}$$

This property is called the scaling property of the fractional Brownian motion. In this section, we are going to study scaling properties of fractional Brownian motions on a Lie group.

As in the previous section, let \mathbf{G} be a connected Lie group (of matrices) with Lie algebra \mathfrak{g} . Let V_1, \dots, V_d be a basis of \mathfrak{g} and denote by $(X_t)_{t \geq 0}$ the solution of the equation

$$dX_t = X_t \left(\sum_{i=1}^d V_i dB_t^i \right), \quad X_0 = 1_{\mathbf{G}},$$

where $(B_t)_{t \geq 0}$ is a d -dimensional fractional Brownian with Hurst parameter $H > \frac{1}{3}$. We restrict to the case $H > \frac{1}{3}$ to use some technical estimates that come from [2] but the results of this section certainly also hold for $H > \frac{1}{4}$.

3.1 Local self-similarity

First, we notice that for $(X_t)_{t \geq 0}$ we always have an asymptotic scaling property in the following sense:

Proposition 3.1 *Let $f : \mathbf{G} \rightarrow \mathbb{R}$ be a smooth map such that $\sum_{i=1}^d (V_i f)(1_{\mathbf{G}})^2 \neq 0$. Then, when $c \rightarrow 0$, $c > 0$, the sequence of processes $(\frac{1}{c^H} (f(X_{ct}) - f(1_{\mathbf{G}})))_{0 \leq t \leq 1}$ converges in law to $(a\beta_t)_{0 \leq t \leq 1}$ where $(\beta_t)_{t \geq 0}$ is a one-dimensional fractional Brownian motion and*

$$a = \sqrt{\sum_{i=1}^d (V_i f)(1_{\mathbf{G}})^2}.$$

Proof. From [2] inequality (4.7),

$$f(X_t) = f(1_{\mathbf{G}}) + \sum_{i=1}^d (V_i f)(1_{\mathbf{G}}) B_t^i + \mathbf{R}(t), \quad t \geq 0$$

for some remainder term \mathbf{R} that satisfies

$$|\mathbf{R}(t)| \leq Ct^{2/p}$$

where C is a random variable with finite exponential moment and $p > 1/H$. Therefore,

$$\frac{1}{c^H} (f(X_{ct}) - f(1_{\mathbf{G}})) = \sum_{i=1}^d (V_i f)(1_{\mathbf{G}}) \frac{B_{ct}^i}{c^H} + \frac{\mathbf{R}(ct)}{c^H}, \quad t \geq 0,$$

and the convergence follows easily from the scaling property of the fractional Brownian motion. \square

Remark 3.2 *Slightly more generally, by using the Taylor expansion proved in [2], we obtain in the same way: Let $f : \mathbf{G} \rightarrow \mathbb{R}$ be a smooth map such that there exist $k \geq 1$ and $(i_1, \dots, i_k) \in \{1, \dots, d\}^k$ that satisfy*

$$(V_{i_1} \cdots V_{i_k} f)(1_{\mathbf{G}}) \neq 0.$$

Denote n the smallest k that satisfies the above property. Then, when $c \rightarrow 0$, $c > 0$, the sequence of processes $(\frac{1}{c^{nH}} (f(X_{ct}) - f(1_{\mathbf{G}})))_{0 \leq t \leq 1}$ converges in law to $(\beta_t)_{0 \leq t \leq 1}$ where $(\beta_t)_{t \geq 0}$ is such that

$$(\beta_{ct})_{t \geq 0} =^{law} (c^{nH} \beta_t)_{t \geq 0}.$$

3.2 Global self-similarity

Despite the local self-similar property, as we will see, in general there is no global scaling property for the fractional Brownian motion on a Lie group. Let us first briefly discuss what should be a good notion of scaling in a Lie group (see also [11] and [12]). If we can find a family a map Δ_c , $c > 0$, such that

$$(X_{ct})_{t \geq 0} =^{law} (\Delta_c X_t)_{t \geq 0},$$

first of all it is natural to require that the map $c \rightarrow \Delta_c$ is continuous and $\lim_{c \rightarrow 0} \Delta_c = 1_{\mathbf{G}}$. Then by looking at $(X_{c_1 c_2 t})_{t \geq 0}$, we will also naturally ask that $\Delta_{c_1 c_2} = \Delta_{c_1} \circ \Delta_{c_2}$. Finally, since $(X_s^{-1} X_{t+s})_{t \geq 0} =^{law} (X_t)_{t \geq 0}$, we will also ask that Δ_c is a Lie group automorphism. The following theorem shows that the existence of such a family Δ_c on \mathbf{G} only holds if the group is $(\mathbf{R}^d, +)$. This is partly due to the following lemma of Lie group theory that says that the existence of a dilation on \mathbf{G} imposes strong topological and algebraic restrictions:

Lemma 3.3 *(See [13]) Assume that there exists a Lie group automorphism $\Psi : \mathbf{G} \rightarrow \mathbf{G}$ such that $d\Psi_{1_{\mathbf{G}}}$ (differential of Ψ at $1_{\mathbf{G}}$) has all its eigenvalues of modulus > 1 , then \mathbf{G} is a simply connected nilpotent Lie group.*

We can now show:

Theorem 3.4 *There exists a family of Lie group automorphisms $\Delta_t : \mathbf{G} \rightarrow \mathbf{G}$, $t > 0$, such that:*

1. *The map $t \rightarrow \Delta_t$ is continuous and $\lim_{t \rightarrow 0} \Delta_t = 1_{\mathbf{G}}$;*
2. *For $t_1, t_2 \geq 0$, $\Delta_{t_1 t_2} = \Delta_{t_1} \circ \Delta_{t_2}$;*
3. *$(X_{ct})_{t \geq 0} =^{law} (\Delta_c X_t)_{t \geq 0}$;*

if and only if the group \mathbf{G} is isomorphic to $(\mathbf{R}^d, +)$.

Proof.

If \mathbf{G} is isomorphic to $(\mathbf{R}^d, +)$, $(X_t)_{t \geq 0}$ is a Euclidean fractional Brownian motion and the result is trivial.

We prove now the converse statement. Let us first show that the existence of the family $(\Delta_t)_{t > 0}$ implies that \mathbf{G} is a simply connected nilpotent Lie group.

Let us denote

$$\delta_c = d\Delta_c(1_{\mathbf{G}}),$$

the differential map of Δ_c at $1_{\mathbf{G}}$ and observe that δ_c is a Lie algebra automorphism $\mathfrak{g} \rightarrow \mathfrak{g}$. The map $f : t \rightarrow \delta_{e^t}$ is a map from \mathbf{R} onto the set of linear maps $\mathfrak{g} \rightarrow \mathfrak{g}$ that is continuous. We have furthermore the property

$$f(t+s) = f(t)f(s).$$

Consequently there exists a linear map $\phi : \mathfrak{g} \rightarrow \mathfrak{g}$ such that

$$\delta_c = \text{Exp}(\phi \ln c), \quad c > 0,$$

where Exp denotes here the exponential of linear maps (and not the exponential map $\mathfrak{g} \rightarrow \mathbf{G}$ which is denoted \exp). Let us furthermore observe that if $\lambda \in \mathbf{Sp}(\phi)$ is an eigenvalue of ϕ , then $\Re \phi > 0$ because $\lim_{c \rightarrow 0} \Delta_c = 1_{\mathbf{G}}$. Therefore from Lemma 3.3, \mathbf{G} has to be a simply connected nilpotent Lie group.

We deduce from Proposition 2.8 that

$$X_t = \exp \left(\sum_{k=1}^{+\infty} \sum_{I=(i_1, \dots, i_k)} \Lambda_I(B)_t V_I \right), \quad t \geq 0,$$

where the above sum is actually finite and

$$\Lambda_I(B)_t = \sum_{\sigma \in \mathfrak{S}_k} \frac{(-1)^{e(\sigma)}}{k^2 \binom{k-1}{e(\sigma)}} \int_{\Delta^k[0,t]} \circ dB^{\sigma^{-1} \cdot I}.$$

Due to the assumption that

$$(X_t)_{t \geq 0} \stackrel{\text{law}}{=} (\Delta_c X_{\frac{t}{c}})_{t \geq 0},$$

we deduce that

$$\left(\exp \left(\sum_{k=1}^{+\infty} \sum_{I=(i_1, \dots, i_k)} \Lambda_I(B)_t V_I \right) \right)_{t \geq 0} \stackrel{\text{law}}{=} \left(\exp \left(\sum_{k=1}^{+\infty} \sum_{I=(i_1, \dots, i_k)} \Lambda_I(B)_{\frac{t}{c}} (\delta_c V)_I \right) \right)_{t \geq 0}.$$

But since the group \mathbf{G} is nilpotent and simply connected the exponential map is a diffeomorphism, therefore

$$\left(\sum_{k=1}^{+\infty} \sum_{I=(i_1, \dots, i_k)} \Lambda_I(B)_t V_I \right)_{t \geq 0} \stackrel{\text{law}}{=} \left(\sum_{k=1}^{+\infty} \sum_{I=(i_1, \dots, i_k)} \Lambda_I(B)_{\frac{t}{c}} (\delta_c V)_I \right)_{t \geq 0}.$$

Let us now observe that due to the scaling property of the fractional Brownian motion

$$\left(\sum_{k=1}^{+\infty} \sum_{I=(i_1, \dots, i_k)} \Lambda_I(B)_{\frac{t}{c}} (\delta_c V)_I \right)_{t \geq 0} =^{law} \left(\sum_{k=1}^{+\infty} \sum_{I=(i_1, \dots, i_k)} \Lambda_I(B)_t \frac{1}{c^{H|I|}} \delta_c V_I \right)_{t \geq 0},$$

where $|I|$ is the length of the word I . Thus, for every $c > 0$,

$$\left(\sum_{k=1}^{+\infty} \sum_{I=(i_1, \dots, i_k)} \Lambda_I(B)_t V_I \right)_{t \geq 0} =^{law} \left(\sum_{k=1}^{+\infty} \frac{1}{c^{kH}} \delta_c \left(\sum_{I=(i_1, \dots, i_k)} \Lambda_I(B)_t V_I \right) \right)_{t \geq 0}$$

Let us now observe that V_1, \dots, V_d is a basis of \mathfrak{g} , therefore all commutators are linear combinations of the V_i 's. By letting $c \rightarrow +\infty$ in the previous equality, we deduce that $c^{-H} \delta_c$ has to be bounded when $c \rightarrow +\infty$. Since $\delta_c = \text{Exp}(\phi \ln c)$, all eigenvalues of ϕ have a real part that is smaller than H . Finally, by letting $c \rightarrow 0$, we conclude that almost surely, for $k \geq 2$,

$$\sum_{I=(i_1, \dots, i_k)} \Lambda_I(B)_t V_I = 0.$$

In particular,

$$\sum_{1 \leq i < j \leq d} \left(\int_0^t B_s^i dB_s^j - B_s^j dB_s^i \right) [V_i, V_j] = 0.$$

Therefore, from the support theorem of [18] (see also [6]), all the brackets have to be 0, that is \mathbf{G} is commutative. \square

Remark 3.5 *The previous theorem in particular applies to the case of a Brownian motion, that corresponds to $H = \frac{1}{2}$. In that case, the theorem can be more easily understood in the following way (see also [11] and [12]). If we denote p_t the density of X_t with respect to the Haar measure; it is well-known that we have an asymptotic development of the form*

$$p_t(1_{\mathbf{G}}) \sim_{t \rightarrow 0} \frac{1}{(2\pi t)^{\frac{d}{2}}} \left(1 + \sum_{k=1}^{+\infty} a_k t^k \right).$$

By homogeneity, the existence of a scaling property for the Brownian motion, would then imply $a_k = 0$ for every k . This implies that the Riemannian curvature of \mathbf{G} is zero, and hence \mathbf{G} is commutative.

If we relax the assumption that the family (V_1, \dots, V_d) forms a basis of the Lie algebra \mathfrak{g} , we can have a global scaling property in slightly more general groups than the commutative ones. Let us first look at one example.

The Heisenberg group \mathbb{H} is the set of 3×3 matrices:

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbf{R}.$$

The Lie algebra of \mathbb{H} is generated by the matrices

$$D_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad D_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

for which the following equalities hold

$$[D_1, D_2] = D_3, \quad [D_1, D_3] = [D_2, D_3] = 0.$$

Consider now the solution of the equation

$$dX_t = X_t(D_1 dB_t^1 + D_2 dB_t^2), \quad X_0 = 1,$$

where (B_t^1, B_t^2) is a two-dimensional fractional Brownian motion with Hurst parameter $H > \frac{1}{4}$. It is easily seen that

$$X_t = \begin{pmatrix} 1 & B_t^1 & \frac{1}{2} \left(B_t^1 B_t^2 + \int_0^t B_s^1 dB_s^2 - B_s^2 dB_s^1 \right) \\ 0 & 1 & B_t^2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore $(X_{ct})_{t \geq 0} =^{law} (\Delta_c X_t)_{t \geq 0}$, where Δ_c is defined by

$$\Delta_c \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & c^H x & c^{2H} z \\ 0 & 1 & c^H y \\ 0 & 0 & 1 \end{pmatrix}.$$

In that case, we thus have a global scaling property whereas \mathbb{H} is of course not commutative but step-two nilpotent. Actually, we shall have a global scaling property in the Lie groups that are called the Carnot groups. Let us recall the definition of a Carnot group.

Definition 3.6 *A Carnot group of step (or depth) N is a simply connected Lie group \mathbb{G} whose Lie algebra can be written*

$$\mathcal{V}_1 \oplus \dots \oplus \mathcal{V}_N,$$

where

$$[\mathcal{V}_i, \mathcal{V}_j] = \mathcal{V}_{i+j}$$

and

$$\mathcal{V}_s = 0, \quad \text{for } s > N.$$

Example 3.7 *Consider the set $\mathbb{H}_n = \mathbb{R}^{2n} \times \mathbb{R}$ endowed with the group law*

$$(x, \alpha) \star (y, \beta) = \left(x + y, \alpha + \beta + \frac{1}{2} \omega(x, y) \right),$$

where ω is the standard symplectic form on \mathbb{R}^{2n} , that is

$$\omega(x, y) = x^t \begin{pmatrix} 0 & -\mathbf{I}_n \\ \mathbf{I}_n & 0 \end{pmatrix} y.$$

On \mathfrak{h}_n the Lie bracket is given by

$$[(x, \alpha), (y, \beta)] = (0, \omega(x, y)),$$

and it is easily seen that

$$\mathfrak{h}_n = \mathcal{V}_1 \oplus \mathcal{V}_2,$$

where $\mathcal{V}_1 = \mathbb{R}^{2n} \times \{0\}$ and $\mathcal{V}_2 = \{0\} \times \mathbb{R}$. Therefore \mathbb{H}_n is a Carnot group of depth 2 and observe that \mathbb{H}_1 is isomorphic to the Heisenberg group.

Notice that the vector space \mathcal{V}_1 , which is called the basis of \mathbb{G} , Lie generates \mathfrak{g} , where \mathfrak{g} denotes the Lie algebra of \mathbb{G} . Since \mathbb{G} is step N nilpotent and simply connected, the exponential map is a diffeomorphism. On \mathfrak{g} we can consider the family of linear operators $\delta_t : \mathfrak{g} \rightarrow \mathfrak{g}$, $t \geq 0$ which act by scalar multiplication t^i on \mathcal{V}_i . These operators are Lie algebra automorphisms due to the grading. The maps δ_t induce Lie group automorphisms $\Delta_t : \mathbb{G} \rightarrow \mathbb{G}$ which are called the canonical dilations of \mathbb{G} . Let us now take a basis U_1, \dots, U_d of the vector space \mathcal{V}_1 . The vectors U_i 's can be seen as left invariant vector fields on \mathbb{G} so that we can consider the following stochastic differential equation on \mathbb{G} :

$$dY_t = \sum_{i=1}^d \int_0^t U_i(Y_s) dB_s^i, \quad t \geq 0, \quad (3.5)$$

which is easily seen to have a unique solution associated with the initial condition $Y_0 = 1_{\mathbb{G}}$. We have then the following global scaling property:

Proposition 3.8

$$(Y_{ct})_{t \geq 0} =^{law} (\Delta_{c^H} Y_t)_{t \geq 0}.$$

Proof. We keep the notations introduced before the proof of Theorem 2.8. From Theorem 2.8, we have

$$Y_t = \exp \left(\sum_{k=1}^N \sum_{I=(i_1, \dots, i_k)} \Lambda_I(B)_t U_I \right), \quad t \geq 0.$$

Therefore,

$$(Y_{ct})_{t \geq 0} =^{law} \left(\exp \left(\sum_{k=1}^N c^{H|I|} \sum_{I=(i_1, \dots, i_k)} \Lambda_I(B)_t U_I \right) \right)_{t \geq 0}.$$

Since

$$\begin{aligned} \exp \left(\sum_{k=1}^N c^{H|I|} \sum_{I=(i_1, \dots, i_k)} \Lambda_I(B)_t U_I \right) &= \exp \left(\sum_{k=1}^N \sum_{I=(i_1, \dots, i_k)} \Lambda_I(B)_t (\delta_{c^H} U_I) \right) \\ &= \Delta_{c^H} \exp \left(\sum_{k=1}^N \sum_{I=(i_1, \dots, i_k)} \Lambda_I(B)_t U_I \right), \end{aligned}$$

we conclude

$$(Y_{ct})_{t \geq 0} =^{law} (\Delta_{c^H} Y_t)_{t \geq 0}.$$

□

The previous proposition admits a counterpart.

Theorem 3.9 *Let \mathbf{G} be a connected Lie group (of matrices) with Lie algebra \mathfrak{g} . Let V_1, \dots, V_d be a family of \mathfrak{g} . Consider now the solution of the equation*

$$dX_t = X_t \left(\sum_{i=1}^d V_i dB_t^i \right), \quad X_0 = 1_{\mathbf{G}}.$$

Assume that there exists a family of Lie group automorphisms $\Delta_t : \mathbf{G} \rightarrow \mathbf{G}$, $t > 0$, such that:

1. *The map $t \rightarrow \Delta_t$ is continuous and $\lim_{t \rightarrow 0} \Delta_t = 1_{\mathbf{G}}$;*
2. *For $t_1, t_2 \geq 0$, $\Delta_{t_1 t_2} = \Delta_{t_1} \circ \Delta_{t_2}$;*
3. *$(X_{ct})_{t \geq 0} =^{law} (\Delta_c X_t)_{t \geq 0}$;*

Then the Lie subgroup \mathbf{H} that is generated by e^{V_1}, \dots, e^{V_d} is a Carnot group.

Proof.

We can follow the lines of the proof of Theorem 3.4 to deduce that \mathbf{H} has to be a simply connected nilpotent group and that for every $c > 0$,

$$\left(\sum_{k=1}^{+\infty} \sum_{I=(i_1, \dots, i_k)} \Lambda_I(B)_t V_I \right)_{t \geq 0} =^{law} \left(\sum_{k=1}^{+\infty} \frac{1}{c^{kH}} \delta_c \left(\sum_{I=(i_1, \dots, i_k)} \Lambda_I(B)_t V_I \right) \right)_{t \geq 0},$$

where δ_c is the differential map of Δ_c at $1_{\mathbf{G}}$. For $k \geq 1$, we denote \mathcal{V}_k the linear space generated by the set of commutators:

$$\{V_I, | I | = k\}.$$

By letting $c \rightarrow +\infty$ and $c \rightarrow 0$ and by using the support theorem of [6], we obtain that $c^{-kH} \delta_c$ is bounded on \mathcal{V}_k for $c \rightarrow +\infty$ and $c \rightarrow 0$. Since $c^{-kH} \delta_c = \text{Exp}((\phi - kH \mathbf{Id}) \ln c)$, for some matrix ϕ , we conclude

$$\mathfrak{h} = \bigoplus_{k=1}^{+\infty} \mathcal{V}_k,$$

where \mathfrak{h} is the Lie algebra of \mathbf{H} . This proves that \mathbf{H} is a Carnot group. □

4 Stochastic analysis of the fractional Brownian motion on a Lie group

As before, let \mathbf{G} be a compact connected Lie group of matrices with Lie algebra \mathfrak{g} . From the theory of compact Lie groups, it is well-known that \mathbf{G} can be endowed with a bi-invariant Riemannian structure. This bi-invariant Riemannian structure induces a scalar product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on \mathfrak{g} for which the maps

$$\begin{aligned} \mathbf{Ad}_g : \mathfrak{g} &\rightarrow \mathfrak{g} \\ x &\rightarrow gxg^{-1}, \end{aligned}$$

are isometries. The map $g \rightarrow \mathbf{Ad}_g$ is called the adjoint representation of \mathbf{G} on \mathfrak{g} . Let V_1, \dots, V_d be a family of \mathfrak{g} that forms an orthonormal basis for the scalar product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$. Consider now the solution of the equation

$$dX_t = X_t \left(\sum_{i=1}^d V_i dB_t^i \right), \quad X_0 = 1_{\mathbf{G}}.$$

where $(B_t)_{t \geq 0}$ is a d -dimensional fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. Our goal in this section is to show that for every $t > 0$ the random variable X_t admits a density with respect to the Haar measure of \mathbf{G} . For that, we shall develop a Malliavin calculus for the fractional Brownian motion on the path group space (see [17] for the Brownian case).

Let us recall (see for instance [8]) that the fractional Brownian motion $(B_t)_{t \geq 0}$ admits the Volterra type representation:

$$B_t^i = \int_0^t \mathbf{K}(t, s) dW_s^i,$$

where $(W_t)_{t \geq 0}$ is a d -dimensional Brownian motion and

$$\mathbf{K}(t, s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du, \quad t > s,$$

where $c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}}$ is the constant such that the covariance function is

$$R(t, s) = \frac{1}{2} (s^{2H} + t^{2H} - |t-s|^{2H}).$$

The key-point of our study is the following integration by parts formula.

Proposition 4.1 *Let $f : \mathbf{G} \rightarrow \mathbb{R}$ and $h : \mathbb{R}_{\geq 0} \rightarrow \mathfrak{g}$ be smooth functions, then we have*

$$\mathbb{E} \left(\left\langle \nabla f(X_t), \int_0^t \left(\int_s^t \frac{\partial \mathbf{K}}{\partial u}(u, s) \mathbf{Ad}_{X_u} du \right) h'(s) ds \right\rangle_{\mathfrak{g}} \right) = \mathbb{E} \left(f(X_t) \int_0^t \left\langle h'(s), \sum_{i=1}^d V_i dW_t^i \right\rangle_{\mathfrak{g}} \right),$$

where $\nabla f = \sum_{i=1}^d (V_i f) V_i$.

Proof. Let $\varepsilon > 0$. We denote

$$W^\varepsilon = W + \varepsilon \int_0^\cdot h'(s)ds.$$

With obvious notations, we have then

$$B^\varepsilon = B + \varepsilon \int_0^\cdot \mathbf{K}(\cdot, s)h'(s)ds.$$

If we consider now the stochastic development on \mathbf{G} of B^ε , that is the solution of the equation

$$dX_t^\varepsilon = X_t^\varepsilon \left(\sum_{i=1}^d V_i dB_t^{i,\varepsilon} \right), \quad X_0^\varepsilon = 1_{\mathbf{G}},$$

we claim that for any smooth $f : \mathbf{G} \rightarrow \mathbb{R}$,

$$\left(\frac{d}{d\varepsilon} \right)_{\varepsilon=0} f(X_t^\varepsilon) = \left\langle \nabla f(X_t), \int_0^t \left(\int_s^t \mathbf{A} d_{X_u} \frac{\partial \mathbf{K}}{\partial u}(u, s) du \right) h'(s) ds \right\rangle_{\mathfrak{g}}, \quad t > 0. \quad (4.6)$$

Indeed let us recall a well-known fact in the theory of linear equations:

Let $(x_t)_{t \geq 0}$ and $(\rho_t)_{t \geq 0}$ be piecewise C^1 and \mathbb{R}^d -valued paths. Denote $(y_t)_{t \geq 0}$ and $(y_t^\varepsilon)_{t \geq 0}$ the respective solutions of the linear equations:

$$dy_t = y_t \left(\sum_{i=1}^d V_i dx_t^i \right), \quad y_0 = 1_{\mathbf{G}},$$

and

$$dy_t^\varepsilon = y_t^\varepsilon \left(\sum_{i=1}^d V_i d(x_t^i + \varepsilon \rho_t^i) \right), \quad y_0^\varepsilon = 1_{\mathbf{G}}.$$

Then we have

$$\left(\frac{dy_t^\varepsilon}{d\varepsilon} \right)_{\varepsilon=0} = \left(\int_0^t \mathbf{A} d_{y_s} \left(\sum_{i=1}^d V_i d\rho_s^i \right) \right) y_t.$$

By applying this result to piecewise linear interpolations of B along the dyadic subdivision and by passing to the limit with the transfer principle of [14] we get formula (4.6)

We deduce therefore our integration by parts formula from the classical integration by parts formula on the Wiener space.

□

Remark 4.2 *As an easy consequence, we can derive a Clark-Ocone type formula for $(X_t)_{t \geq 0}$: If $f : \mathbf{G} \rightarrow \mathbb{R}$ is a smooth function, then we have*

$$f(X_t) = \mathbb{E}(f(X_t)) + \int_0^t \int_s^t \left\langle \frac{\partial \mathbf{K}}{\partial u}(u, s) \mathbb{E}((\mathbf{A} d_{X_u})^* \nabla f(X_t) \mid \mathcal{F}_s) du, \sum_{i=1}^d V_i dW_s^i \right\rangle_{\mathfrak{g}}.$$

Remark 4.3 More generally, let (\mathbb{M}, g) be a d -dimensional compact connected Riemannian manifold. We denote $\mathcal{O}(\mathbb{M})$ the orthonormal frame bundle of \mathbb{M} and π the canonical surjection on \mathbb{M} . Let $(H_i)_{i=1, \dots, d}$ the horizontal fundamental vector fields of $\mathcal{O}(\mathbb{M})$ and $\overline{\Omega}$ the equivariant representation of the Cartan curvature form Ω . Consider the rough paths differential equation

$$dX_t^* = \sum_{i=1}^d H_i(X_t^*) dB_t^i, \quad X_0^* = m^* \in \mathcal{O}(\mathbb{M}),$$

Due to the compactness of \mathbb{M} , from rough paths theory [15] and [7], it is easy to show that the previous equation has a unique solution. By following the Eels-Elworthy-Malliavin's approach to Riemannian Brownian motion (see [17]), one can define a fractional Brownian motion $(X_t)_{t \geq 0}$ on \mathbb{M} by:

$$(X_t)_{t \geq 0} = (\pi X_t^*)_{t \geq 0}.$$

In this setting, by using Malliavin-Cruzeiro isomorphism (see [17]), we can show the following integration by parts formula. Let $h = (h_1, \dots, h_d) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^d$ be a smooth function. For any smooth $\xi : \mathbb{M} \rightarrow \mathbb{R}$,

$$\mathbb{E} \left(\langle \nabla \xi(X_t), X_t^* \Theta_t \rangle_{T_{X_t} \mathbb{M}} \right) = \mathbb{E} \left(\xi(X_t) \sum_{i=1}^d \int_0^t h'_i(s) dW_s^i \right), \quad t > 0,$$

where Θ is the \mathbb{R}^d -valued process solution of the rough paths differential equation

$$\Theta_t = \int_0^t \mathbf{K}(t, s) h'(s) ds + \int_0^t \int_0^s \overline{\Omega}_{X_u^*}(\Theta_u, dB_u) \cdot dB_s.$$

As a corollary of the previous integration by parts formula, we deduce:

Proposition 4.4 For every $t > 0$ the random variable X_t admits a density with respect to the Haar measure of \mathbf{G} .

Proof. We have to show that for $t > 0$, the Malliavin matrix Γ_t is invertible. Here, we have from the integration by parts formula:

$$\Gamma_t = \int_0^t \left(\int_s^t \frac{\partial \mathbf{K}}{\partial u}(u, s) \mathbf{Ad}_{X_u} du \right)^* \left(\int_s^t \frac{\partial \mathbf{K}}{\partial u}(u, s) \mathbf{Ad}_{X_u} du \right) ds$$

If $x \in \mathbf{Ker} \Gamma_t$, then we have

$$x^* \Gamma_t x = 0.$$

It implies that for $s \leq t$,

$$\int_s^t \frac{\partial \mathbf{K}}{\partial u}(u, s) \mathbf{Ad}_{X_u}(x) du = 0,$$

and so for $s \leq t$,

$$\mathbf{Ad}_{X_s}(x) = 0.$$

Since \mathbf{Ad}_{X_s} is an isometry, we get therefore $x = 0$. □

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